Precalculus Exam - University of Houston Math Contest
January 30, 2021

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1) Given: $f(x) = 3e^{2x}$ and $g(x) = \log_5(4x+1)$, evaluate $(f^{-1} \circ g^{-1})(2)$.

a) $\ln(2)$ b) $\ln(\sqrt{2})$ c) $\ln(3)$ d) $\ln(4)$ e) $2\ln(2)$ f) $\ln\left(\frac{1}{2}\right)$ g) None of the above

2) Let $f(x^2+1) = x^4 + 3x^2 + 5$; find $f(-1) + f(-5) + 4f\left(-\frac{1}{2}\right) = ?$

a) 42 b) 34 c) 16 d) 40 e) 21 f) 37 g) None of the above

3) Order the following numbers from smallest to greatest.

$a = \sin\left(\frac{7\pi}{3}\right) + \cos\left(\frac{35\pi}{6}\right)$ $b = \sin\left(\frac{31\pi}{6}\right) + \cos\left(\frac{49\pi}{3}\right)$ $c = \sin\left(\frac{21\pi}{4}\right) + \cos\left(\frac{61\pi}{4}\right)$

$d = \sin\left(\frac{43\pi}{4}\right) + \sec\left(\frac{83\pi}{4}\right)$ $e = \tan\left(\frac{85\pi}{4}\right) + \cos\left(\frac{32\pi}{3}\right)$

a) $c < d < b < a < e$ b) $d < c < b < e < a$ c) $c < d < e < b < a$ d) $d < c < e < b < a$
e) $d < c < b < a < e$ f) $c < d < b < e < a$ g) None of the above

4) Given: $m - n = 3$ and $n - k = 5$, let $a = 5m^2 - 8n^2 + 3k^2$.
If $f(x) = 2x\cos(\pi x) + \sin\left(\frac{\pi}{2}x\right)$, find the value of $f(a)$.

a) 480 b) 120 c) 600 d) 300 e) 360 f) 240 g) None of the above

5) Given $f(x) = 120 - 5\sin(\pi x + 1)$ and $g(x) = 20\tan(4x - 2)$; let P_1 and P_2 be the periods for the functions $f(x)$ and $g(x)$, respectively.
Find the value of: $\sin\left(\frac{P_2}{P_1}\right)\cos\left(\frac{P_2}{P_1}\right)$

a) $\frac{\sqrt{6}}{4}$ b) $\frac{\sqrt{2}}{2}$ c) $\frac{\sqrt{2}}{4}$ d) $\frac{\sqrt{3}}{4}$ e) $\frac{\sqrt{6}}{2}$ f) $2\sqrt{2}$ g) None of the above

6) The following function is graphed over the interval $[0, \pi)$: $f(x) = 2\sin^2(4x) - 3\sin(4x) + 1$
Find the sum of the smallest and the largest *x*-intercepts of this function over this interval.

a) $\frac{3\pi}{4}$ b) $\frac{11\pi}{24}$ c) $\frac{13\pi}{24}$ d) $\frac{3\pi}{8}$ e) $\frac{19\pi}{24}$ f) $\frac{\pi}{4}$ g) None of the above

7) Given: $x^x = \dfrac{1}{\sqrt[9]{3}}$, find the value of: $81x^2 + \dfrac{1}{3x} = ?$

a) $\dfrac{730}{27}$ b) $\dfrac{26}{3}$ c) $\dfrac{89}{9}$ d) $\dfrac{82}{9}$ e) $\dfrac{28}{9}$ f) $\dfrac{28}{3}$ g) None of the above

8) Given: $(1+\sin t)(1+\cos t) = \dfrac{5}{4}$, find the value of $(1-\sin t)(1-\cos t) = ?$

a) $\dfrac{13+4\sqrt{10}}{4}$ b) $\dfrac{13-4\sqrt{10}}{4}$ c) $\dfrac{13-\sqrt{10}}{4}$ d) $\dfrac{11+2\sqrt{10}}{4}$

e) $\dfrac{11-2\sqrt{10}}{4}$ f) $\dfrac{15-\sqrt{10}}{4}$ g) None of the above

9) Let: $x = \sqrt{\dfrac{2-\sqrt{2}}{2+\sqrt{2}}} + \dfrac{2\sqrt{2}-2}{\sqrt{2}}$ and $y = \sqrt{\dfrac{3+\sqrt{3}}{3-\sqrt{3}}}$

Which of the following is equivalent to $\dfrac{x}{y}$?

a) $\sqrt{4-\sqrt{3}}$ b) $\sqrt{1-\sqrt{3}}$ c) $\sqrt{2+\sqrt{3}}$ d) $\sqrt{4-2\sqrt{3}}$

e) $\sqrt{2-\sqrt{3}}$ f) $\sqrt{2\sqrt{3}-2}$ g) None of the above

10) Let S be the set of all integers that are in the domain of the following function:

$$f(x) = \frac{\arcsin\left(\frac{x^2-2x}{8}\right)}{\log_6\left(2x^2-4x\right)}$$

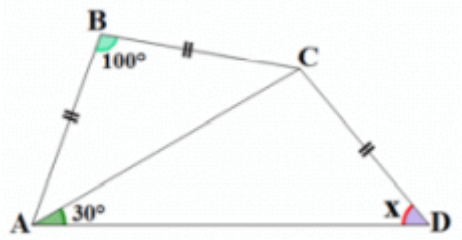
Find the sum of the absolute values of all elements in S. That is, find $\sum_{k\in S} |k|$.

a) 10 b) 3 c) 9 d) 8 e) 13 f) 11 g) None of the above

11) Given: $\dfrac{x^{-x}(x^2-2x)+x^{-x+2}}{x-x^2} = -\dfrac{1}{128}$. Find the remainder when $5x$ is divided by 7.

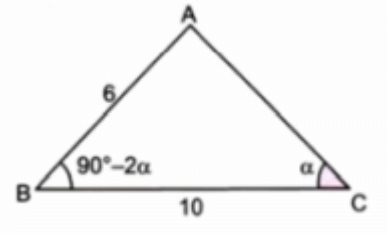
a) 6 b) 5 c) 3 d) 4 e) 1 f) 2 g) None of the above

12) Given: $AB = BC = CD$, $m(\angle ABC) = 100^\circ$, $m(\angle DAC) = 30^\circ$ and $m(\angle CDA) = x$.
Find the value of: $6\sin(3x)\tan(6x) = ?$ *(Note: The image may not be drawn to scale.)*



a) $2\sqrt{3}$ b) $-3\sqrt{3}$ c) $3\sqrt{3}$ d) $-6\sqrt{3}$ e) $-\sqrt{3}$ f) $-2\sqrt{3}$ g) None of the above

13) For $\triangle ABC$, we have the following information: $AB = 6$, $BC = 10$, $m(\angle ACB) = \alpha$, $m(\angle ABC) = 90 - 2\alpha$. Find the value of: $34\sin(4\alpha)$. *(Note: The image may not be drawn to scale.)*



a) $\dfrac{17}{45}$ b) $\dfrac{240}{17}$ c) $\dfrac{320}{17}$ d) 120 e) $\dfrac{480}{17}$ f) 240 g) None of the above

14) Assume that $\tan(3^\circ) = a$. Write the value of the following expression in terms of a:

$$2\left(\frac{\tan(3^\circ)\tan(31^\circ)}{\tan(1^\circ)\tan(61^\circ)}\right)\left(\frac{\tan(63^\circ)+\tan(33^\circ)}{1-\tan(63^\circ)\tan(33^\circ)}\right)$$

a) $1 - \dfrac{1}{a}$ b) $\dfrac{1}{a} - a$ c) $a - \dfrac{1}{a}$ d) $a + \dfrac{1}{a}$ e) $\dfrac{a}{a^2 - 1}$ f) None of the above

15) Given $\cos(4x) = \dfrac{1}{10}$, find the value of the following expression:

$$\frac{\left(\sin^8(x) - \cos^8(x)\right)}{2\cos(2x)} = ?$$

a) $\dfrac{31}{40}$ b) $\dfrac{31}{80}$ c) $-\dfrac{31}{80}$ d) $-\dfrac{51}{80}$ e) $\dfrac{51}{80}$ f) $-\dfrac{21}{40}$ g) None of the above

16) Consider the functions: $f(x) = \dfrac{1-\cos(x)}{2\sin^2(x) - \sin(x) - 1}$ and $g(x) = \dfrac{1-\sin(x)}{4\sin(x)\cos^2(x) - 3\sin(x)}$.

Find the smallest positive real number that is not in the domain of either of these functions.

a) $\dfrac{2\pi}{3}$ b) $\dfrac{5\pi}{6}$ c) $\dfrac{\pi}{6}$ d) $\dfrac{\pi}{3}$ e) $\dfrac{11\pi}{6}$ f) $\dfrac{7\pi}{6}$ g) None of the above

17) Let $K = \cos\left(2\arcsin\left(\frac{1}{3}\right)\right),\ L = \sin\left(2\arccos\left(\frac{1}{4}\right)\right),\ M = \cot\left(2\arctan\left(\frac{1}{2}\right)\right).$

Find the value of: $K \cdot L \cdot M$.

a) $\frac{7\sqrt{15}}{96}$ b) $\frac{7\sqrt{15}}{48}$ c) $\frac{\sqrt{15}}{288}$ d) $\frac{14\sqrt{15}}{45}$ e) $\frac{5\sqrt{10}}{96}$ f) $\frac{7\sqrt{10}}{288}$ g) None of the above

18) Consider the following functions over the interval $[0, 2\pi]$. At how many points do the graphs of these functions intersect?

$$f(x) = 8\sin^2(3x) - \sin(3x)$$
$$g(x) = 5 - 5\cos^2(3x)$$

a) 16 b) 13 c) 12 d) 9 e) 5 f) 11 g) None of the above

19) In triangle ABC, we have the following relations:

$$3\sin(\angle A) - 4\sin(\angle B) = \sin(\angle C) \text{ and } 3a - 4b = 20 - c$$

Find the side length c.

a) 10 b) 5 c) 8 d) 6 e) 12 f) 4 g) None of the above

20) For $\triangle ABC$, we have the following information regarding side lengths:

$$b^3 - c^3 = ba^2 - ca^2 \text{ and } b \neq c$$

Find the measure of the angle A.

a) 30° b) 150° c) 45° d) 90° e) 120° f) 60° g) None of the above

21) Assume that $\tan(10^\circ) = a$ and $\sec(10^\circ) = b$.

Which of the following would be equivalent to the expression: $\dfrac{\sin(20^\circ)\cos(20^\circ)}{\cos^2(80^\circ) + \sin(50^\circ) + \cos^2(10^\circ)}$?

a) $\frac{a}{2 - b^2}$ b) $\frac{a + b}{2 - b^2}$ c) $\frac{2a}{2 - b^2}$ d) $\frac{b^2}{a}$ e) $\frac{2a}{b^2}$ f) $\frac{a}{b^2}$ g) None of the above

22) Let $y = \arccos\left(\frac{x}{2}\right)$ where $0 < x \leq 0.1$. Which of the following is an expression for $2\cos(3y)$ in terms of x?

a) $\frac{5x - 2x^3}{2}$ b) $\frac{x\sqrt{4 - x^2}}{2 - x^2}$ c) $\frac{2x - x^3}{4}$ d) $-2x^3 + 6x$

e) $\frac{x\sqrt{4 - x^2}}{2}$ f) $x^3 - 3x$ g) None of the above

23) Let $f(x) = 13 + 6\sin(x) - 9\cos^2(x)$. What is the maximum value of this function?

a) 19 b) 17 c) 14 d) 15 e) 12 f) 22 g) None of the above

24) Find the domain of the following function: $f(x) = 2\arccos\left(\frac{5x}{6} - 3\right) + 1.$

a) $[-1, 3]$ b) $\left[\frac{12}{5}, \frac{19}{5}\right]$ c) $\left[\frac{17}{6}, \frac{9}{6}\right]$ d) $\left[\frac{19}{6}, \frac{24}{5}\right]$

e) $\left[\frac{12}{5}, \frac{24}{5}\right]$ f) $\left[\frac{1}{5}, \frac{7}{5}\right]$ g) None of the above

25) For an arithmetic sequence $\{a_k\}$, let $S_n = \sum_{k=1}^{n} a_k$ denote the sum of the first n terms of this sequence. Given: $S_5 + S_6 = 44$ and $S_{11} = 47$, find S_{22}.

a) 106.1 b) 112.1 c) 108.2 d) 98.2 e) 102.1 f) 95.3 g) None of the above

26) Given: $\tan(x) - \cot(x) = 1$, find the value of: $\tan^6(x) + \cot^6(x) = ?$

a) 16 b) 4 c) 8 d) 18 e) 12 f) 10 g) None of the above

27) Given: $\sin(x) - \cos(x) = \frac{1}{\sqrt{5}}$; find the value of $\tan(x) + \cot(x) + \cos(2x)$.

a) $\frac{22}{5}$ b) $\frac{13}{5}$ c) $\frac{31}{10}$ d) $\frac{23}{10}$ e) $\frac{33}{5}$ f) $\frac{33}{10}$ g) None of the above

28) If $f(\cos x) = \cot x$ and $0 < x < 0.2$, find a formula for $f(2x)$.

a) $\frac{x}{\sqrt{1-x^2}}$ b) $\frac{2}{\sqrt{1-4x^2}}$ c) $\frac{x}{\sqrt{1+x^2}}$ d) $\frac{2x}{\sqrt{1-x^2}}$

e) $\frac{2x}{\sqrt{1+4x^2}}$ f) $\frac{2x}{\sqrt{1-4x^2}}$ g) None of the above

29) Given: $\dfrac{1-\cos(2\theta)}{\sin(2\theta)} = \cot\left(\dfrac{\pi}{10}\right)$ and $\theta > 0$; find the sum of the two smallest possible values for θ.

a) $\dfrac{9\pi}{10}$ b) $\dfrac{11\pi}{10}$ c) $\dfrac{19\pi}{10}$ d) $\dfrac{6\pi}{5}$ e) $\dfrac{13\pi}{10}$ f) $\dfrac{9\pi}{5}$ g) None of the above

30) Consider the following equation over the interval $(\pi, 2\pi)$: $(\tan(x)+\cot(x))^2 = 16$
Find the sum of the two largest solutions in this interval.

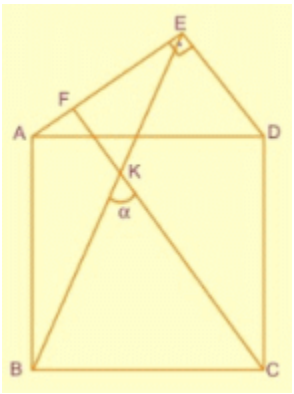
a) $\dfrac{5\pi}{2}$ b) $\dfrac{41\pi}{12}$ c) $\dfrac{10\pi}{3}$ d) $\dfrac{31\pi}{12}$ e) $\dfrac{11\pi}{6}$ f) $\dfrac{7\pi}{2}$ g) None of the above

31) How many solutions does the following system have?

$$25x^2 + 16y^2 - 200x - 32y + 16 = 0$$
$$4y^2 - x^2 = 16$$

a) 4 b) 3 c) 1 d) 2 e) 6 f) 0 g) None of the above

32) Given: $ABCD$ is a square, $m(\angle EDA) = m(\angle BCF)$, $m(\angle BKC) = \alpha$, $ED = 1$, and $FC = 3$. Find: $\tan(\alpha) = ?$ *(Note: The image may not be drawn to scale.)*



a) $\dfrac{3}{4}$ b) $\dfrac{7}{5}$ c) $\dfrac{5}{3}$ d) $\dfrac{3}{2}$ e) $\dfrac{1}{3}$ f) $\dfrac{4}{3}$ g) None of the above

33) Consider the equation: $\dfrac{\sqrt{3}}{2}\sin(3x) = 1 + \dfrac{1}{2}\cos(3x)$. Find the sum of all solutions of this equation over the interval $[2\pi, 3\pi]$.

a) $\dfrac{23\pi}{3}$ b) $\dfrac{46\pi}{9}$ c) $\dfrac{26\pi}{9}$ d) $\dfrac{13\pi}{3}$ e) $\dfrac{23\pi}{6}$ f) $\dfrac{34\pi}{9}$ g) None of the above

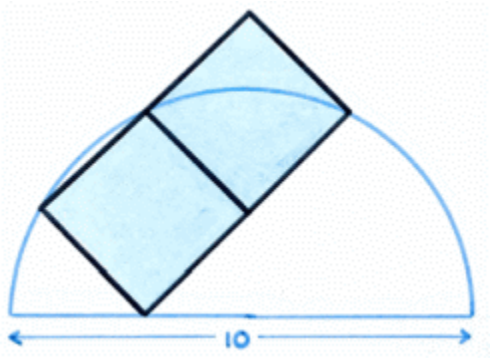
34) If x is an acute angle satisfying: $\dfrac{\sin(5x)}{\sin(2x)\sin(3x)} - \cot(2x) = \sqrt{3}$; find the sum of all possible values of x.

a) $\dfrac{5\pi}{18}$ b) $\dfrac{2\pi}{9}$ c) $\dfrac{4\pi}{9}$ d) $\dfrac{5\pi}{9}$ e) $\dfrac{3\pi}{20}$ f) $\dfrac{\pi}{18}$ g) None of the above

35) Consider the equation: $\sin(5x)\cos(3x) = \sin(9x)\cos(7x)$. How many solutions does this equation have over the interval $[0, 2\pi]$?

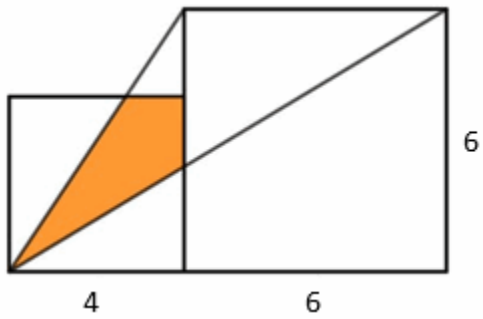
a) 33 b) 13 c) 32 d) 34 e) 28 f) 26 g) None of the above

36) The diameter of the semi-circle on the image below is 10. What is the total area of the two squares on the figure below? *(Note: The image may not be drawn to scale.)*



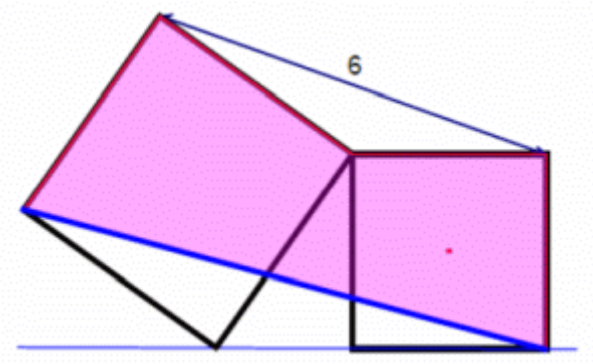
a) 25 b) 24 c) 10 d) 15 e) 20 f) 40 g) None of the above

37) The following figure has two squares; one with side length 4 and the other with side length 6. What is the area of the shaded region? *(Note: The image may not be drawn to scale.)*



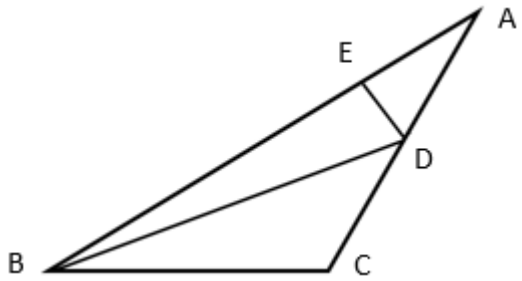
a) $\dfrac{81}{15}$ b) $\dfrac{88}{15}$ c) $\dfrac{55}{10}$ d) $\dfrac{57}{10}$ e) $\dfrac{91}{15}$ f) $\dfrac{61}{10}$ g) None of the above

38) There are two squares (non-identical) on the figure below. Find the area of the shaded region. *(Note: The image may not be drawn to scale.)*



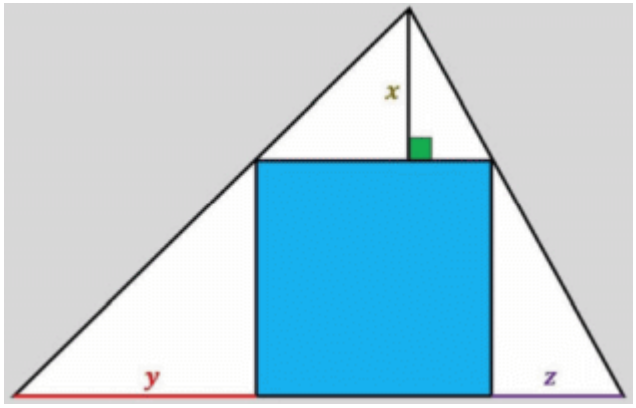
a) 16 b) 18 c) 21 d) 20 e) 24 f) 27 g) None of the above

39) Given: $m(\angle EBD) = m(\angle DBC) = 15^\circ,\ m(\angle BED) = 90^\circ,\ ED = 2,\ BC = 6.$
Find the area of the triangle ABC. *(Note: The image may not be drawn to scale.)*



a) 27 b) 24 c) 16 d) 36 e) 20 f) 18 g) None of the above

40) The figure below contains a square inscribed in a triangle. Given: $x = 2,\ y = 3.1,\ z = 1.2.$
Find the area of the square. *(Note: The image may not be drawn to scale.)*



a) $\frac{41}{6}$ b) $\frac{43}{5}$ c) $\frac{33}{4}$ d) $\frac{79}{10}$ e) $\frac{81}{10}$ f) $\frac{38}{5}$ g) None of the above