Algebra 2 Exam - University of Houston Math Contest
January 29, 2022

1) Find the inverse of $y = 2x - 3$.

a) $y = \dfrac{x}{2} + 3$ b) $y = \dfrac{x+3}{2}$ c) $y = 2x + 3$

d) $y = \dfrac{1}{2x-3}$ e) $y = \dfrac{1}{2}x - \dfrac{1}{3}$ f) None of these

2) If $f(x) = x^2 + 1$ and $g(x) = 5x - 3$, find $(f \circ g)(x)$.

a) $5x^2 + 2$ b) $5x^2 + 5$ c) $25x^2 + 10$

d) $25x^2 - 30x + 10$ e) $25x^2 - 30x + 9$ f) None of these

3) Solve for $x : \ |3x + 7| = 5$.

a) $x = -\dfrac{2}{3}$ b) $x = -\dfrac{2}{3}, \ x = -4$ c) $x = \dfrac{2}{3}, \ x = 4$

d) $x = \dfrac{2}{3}$ e) $x = -\dfrac{2}{3}, \ x = \dfrac{2}{3}$ f) None of these

4) Simplify the following: $(4 - 3i)(5 + 8i)$.

a) 27 b) $20 + 24i$ c) $44 + 17i$ d) $-4 + 17i$ e) $20 + 41i$ f) None of these

5) It takes Mary 20 minutes to walk uphill to her school which is 1 mile away, but it only takes her 10 minutes to walk downhill from her school to her home along the same route. Her sister Ally bikes with an average speed that is twice Mary's overall average speed on this trip. If Ally bikes at this speed to the library which is 4 miles away from their home; how long does it take for Ally to reach the library?

a) 20 minutes   b) 30 minutes   c) 15 minutes

d) 60 minutes   e) 40 minutes   f) None of these

6) The variable A varies directly as the cube of B, and B varies inversely as the square root of C. If A equals $1/2$ when C equals 400, what is the value of C when A equals 100?

a) $8\sqrt[3]{5}$ b) $4\sqrt[3]{25}$ c) $20\sqrt{5}$ d) $2\sqrt[3]{5}$ e) $16\sqrt[3]{5}$ f) None of these

7) A, B and C are three non-empty sets. $1/4$ of all elements of A are also elements of B, and $1/5$ of all elements of B are also elements of A. C has no elements that are common with A; $1/2$ of all elements of C are also elements of B.
If A has a total of 255 proper subsets, find the number of elements in the set: $(A \cup B) \cup (A \cap C)$.

a) 64   b) 32   c) 24   d) 16   e) 72   f) None of these

8) Given: $f(x) = 5e^{x+1} - \log_{\sqrt{6}}(x+3)$; $g(x) = 2^{x-3} + 5$; and $h(x) = 5\ln(x-2)$. Evaluate: $h^{-1}(20) + f(g^{-1}(6))$.

a) $e^6 - 2$ b) $6e^6 - 4$ c) $6e^4$ d) $4e^4 - 4$ e) $6e^4 - 4$ f) None of these

9) Consider three numbers x, y and z such that: $505x + 606y = 606$ and $505z - 101y = 404$. Let A be the average of the numbers. Find the value of: $111A$.

a) 148   b) 74   c) 54   d) 37   e) 81   f) None of these

10) Let x, y and z be three real numbers. The average of these three numbers is 6 and the average of their reciprocals is $\dfrac{5}{36}$. If the product of these numbers is 24, what is the sum of the squares of these numbers?

a) 108   b) 124   c) 304   d) 324   e) 332   f) 248   g) None of these

11) Of the 600 residents in a town, 30% watch the TV Show A, 40% watch the TV show B, and 50% watch the TV show C. If all residents watch at least one of these three shows, and 10% watch exactly two of these three shows, then how many people watch all of the shows?

a) 25   b) 24   c) 30   d) 60   e) 45   f) None of these

12) The operation $*$ is defined as: $a * b = 22^a + 9^b$. When the number $2022 * M$ is divided by 11, the remainder is 4. Given M is a positive integer, find the smallest possible value for M.

a) 2   b) 3   c) 4   d) 5   e) 1   f) 0   g) None of these

13) Suppose that a cows give b gallons of milk in c days. At this rate, how many gallons of milk will d cows give in e days?

a) $\dfrac{abc}{de}$ b) $\dfrac{ac}{bde}$ c) $\dfrac{bde}{ac}$ d) $\dfrac{abde}{c}$ e) $\dfrac{bcde}{a}$ f) None of these

14) Suppose that the real number x satisfies

$$\sqrt{49 - x^2} - \sqrt{25 - x^2} = 3.$$

Find the value of $\sqrt{49 - x^2} + \sqrt{25 - x^2}$.

a) 12 b) 9 c) 8 d) $\sqrt{33} + 8$ e) $2\sqrt{10} + 4$ f) None of these

15) What is the remainder when $x^{2022} + 2021x^{2021} + 2020$ is divided by $x + 1$?

a) 3 b) 0 c) 4042 d) 1 e) 2019 f) None of these

16) Find the value of a for which the system of equations

$$3x + 2y = 8$$
$$ax - 8y = 9$$

has no solutions (x, y).

a) 1 b) 0 c) 3 d) -12 e) -8

17) Which of the following describes the set of values of a for which the curves $x^2 + y^2 = a^2$ and $y = x^2 - a$ in the real xy-plane intersect at exactly 3 points?

a) $a > \dfrac{1}{4}$ b) $a > \dfrac{1}{2}$ c) $a = \dfrac{1}{2}$ d) $a = \dfrac{1}{4}$ e) $\dfrac{1}{4} < a < \dfrac{1}{2}$

18) Let $f(x) = \dfrac{x - 1}{x + 1}$ and let $f^{(n)}(x) = f(f(f\ldots(f(x))))$ be the n-fold composition of $f(x)$ with itself. That is $f^{(1)}(x) = f(x)$ and $f^{(n)}(x) = f(f^{(n-1)}(x))$.

Calculate $f^{(2022)}(x)$.

a) $-\dfrac{1}{x}$ b) $-\dfrac{x + 1}{x - 1}$ c) x d) $-\dfrac{x - 1}{x + 1}$ e) $\dfrac{x - 1}{x + 1}$

19) Suppose $g(x) = e^x$, $h(x) = 10^x$, and $f(x) = g(m(x)) = h(n(x))$ for all real numbers x. How many of the following statements are true concerning the intersection points of the graphs of $m(x)$ and $n(x)$?

- The set of all points of the form $(a, m(a))$ where $m(a) = \dfrac{n(a)}{\log_{10} e}$.

- Any coordinate point of $(a, n(a))$ where $n(a) = \dfrac{m(a)}{\ln 10}$.

- The functions have no intersection points.
- Every point on the graphs is an intersection point because the two graphs are the same.

a) 3 b) 1 c) 0 d) 4 e) 2 f) None of the above.

20) In the figure below, n congruent semicircles lie on the diameter of a large semicircle, with their diameters covering the diameter of the large semicircle with no overlap. Let A be the combined area of the small semicircles and B be the area of the region inside the large semicircle but outside the semicircles. The ratio $A : B$ is $1 : 18$.
Find n.
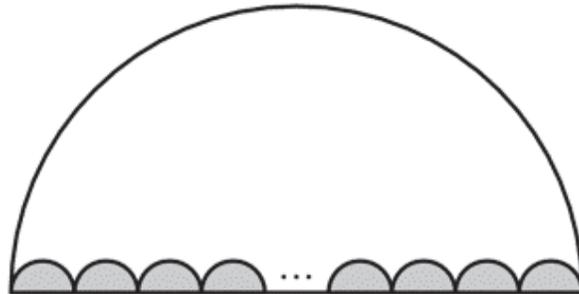
a) 19　　b) 16　　c) 17　　d) 36　　e) 18

21) Sara makes a staircase out of toothpicks as shown:
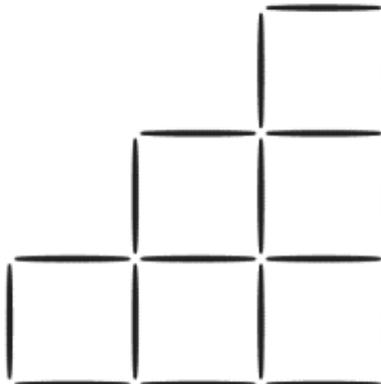
This is a 3-step staircase and uses 18 toothpicks. How many steps would be in a staircase that used 180 toothpicks?

a) 10　　b) 12　　c) 11　　d) 24　　e) 30

22) A box contains 28 red balls, 20 green balls, 19 yellow balls, 13 blue balls, 11 white balls, and 9 black balls. What is the minimum number of balls that must be drawn from the box without replacement to guarantee that at least 15 balls of a single color will be drawn?

a) 91　　b) 84　　c) 75　　d) 79　　e) 76

23) At a certain High School, 60% of the students like dancing, and the rest dislike it. Of those who like dancing, 80% say that they like it, and the rest say that they dislike it. Of those who dislike dancing, 90% say that they dislike it, and the rest say that they like it. What fraction of students who say they dislike dancing actually like it?

a) 10%   b) 20%   c) 25%   d) 12%   e) 33.3%

24) There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

a) 4   b) 3   c) 1   d) 2   e) 5

25) Last year Isabella took 7 math tests and received 7 different scores, each an integer between 91 and 100, inclusive. After each test she noticed that the average of her test scores was an integer. Her score on the seventh test was 95. What was her score on the sixth test?

a) 96   b) 92   c) 98   d) 94   e) 100

26) The graph of the function $y = \dfrac{x-3}{x^2 - x - 6}$ is obtained from the graph of the function $y = \dfrac{1}{x+2}$ by deleting a single point (u, v). What is the product $u \cdot v$?

a) $-\dfrac{1}{5}$ b) $\dfrac{1}{5}$ c) 0   d) $\dfrac{3}{5}$ e) $-\dfrac{3}{5}$

27) Find the value of the expression

$$S = 1! \cdot 3 - 2! \cdot 4 + 3! \cdot 5 - 4! \cdot 6 + \ldots - 2020! \cdot 2022 + 2021! \cdot 2023 - 2022!$$

a) 2021   b) 2022   c) -1 d) -2022 e) 1

28) The polynomial $x^4 + 2x^3 + ax^2 + 5x + b$ is divisible by $x^2 + x - 6$ for certain values of a and b. What is the sum $a + b$?

a) 72   b) 58   c) -72 d) -60 e) -54

29) The graph of which of the following functions has a horizontal asymptote at $y = 2$?

a) $f(x) = \ln(x) + 2$ b) 2^{x-2} c) $f(x) = \dfrac{1}{x} + 2$ d) e^{x+2} e) $\dfrac{1}{x-2}$

30) Consider four integers R, S, T, and U. The mean of these integers is 6, their median is 5.5, their mode is 3, and their range is 7. What is the product RSTU of the four integers?

a) 567   b) 693   c) 735   d) 792   e) 720

31) Find the maximum value of the function $f(x) = \dfrac{10}{4x^2 + 12x + 13}$

a) $\dfrac{5}{6}$ b) $\dfrac{1}{4}$ c) $\dfrac{5}{2}$ d) $\dfrac{10}{13}$ e) 2
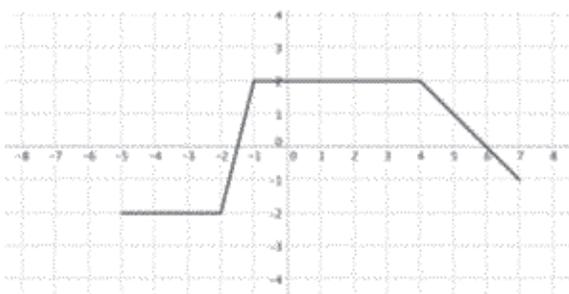
32) Let $f(x) = \log_2(x)$. Which of the following is equal to $f(x^2) - f(4x) + f(8)$?

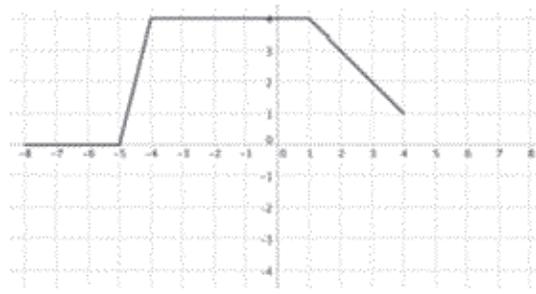a) $f(x) - 5$ b) $f(x)^2 - 4f(x) + 3$ c) $f(x) + 1$ d) $3f(x) + 1$ e) $f(2x - 4)$

33) Find the range of the inverse function $f^{(-1)}(x)$ if $f(x) = \dfrac{x - 3}{2x - 8}$

a) $\left(-\infty, -\dfrac{3}{2}\right) \cup \left(-\dfrac{3}{2}, \infty\right)$ b) $(-\infty, 4) \cup (4, \infty)$ c) $(-\infty, 3) \cup (3, \infty)$

d) $\left(-\infty, \dfrac{3}{8}\right) \cup \left(\dfrac{3}{8}, \infty\right)$ e) $\left(-\infty, \dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, \infty\right)$

34) The graphs of $f(x)$ and $g(x)$ are shown below. Express $g(x)$ in terms of $f(x)$.

The graph of $f(x)$ The graph of $g(x)$

a) $g(x) = f(x + 3) + 2$ b) $g(x) = f(x + 5)$ c) $g(x) = f(x - 3) + 2$

d) $g(x) = f(x + 2) + 3$ e) $g(x) = f(x - 2) + 3$

35) Three consecutive integers are each no more than a distance of 5 units from 3. Their sum is at least a distance of 10 units from 4. One of these integers is:

a) 0 b) 6 c) 10 d) -4 e) -2

36) A Pythagorean Triple is defined as three positive integers a, b and c which satisfy the Pythagorean Theorem, $a^2 + b^2 = c^2$. Assuming positive integer values, three of the following answer choices form a Pythagorean Triple. Select the answer choice which does not belong to this Pythagorean Triple.

a) $m^2 + n^2$ b) $(m - n)^2$ c) $2mn$ d) $m^2 - n^2$

37) Consider three integers such that the sum of the smallest two is 11, the sum of the largest two is 17, and the sum of the largest and the smallest is 13. What is the average of these three integers?

a) 4.333 b) 6.333 c) 8.333 d) 7.666 e) 5.666 f) None of the above

38) Find all real solutions to the following equation:

$$\sqrt{20x^3 + 29x^2 - 100x + 4} = 2x^2 + 5x - 10$$

a) $\{-\sqrt{3} + 2\sqrt{2}, -\sqrt{3} - 2\sqrt{2}\}$ b) $\{\sqrt{3} + 2\sqrt{2}, \sqrt{3} - 2\sqrt{2}\}$ c) $\{-\sqrt{3}, 2\sqrt{2}\}$

d) $\{\sqrt{3}, -2\sqrt{2}\}$ e) $\{\sqrt{3}, 2\sqrt{2}\}$ f) None of the other answers provided.

39) Consider a cubic function, $f(x) = ax^3 + bx^2 + cx + d$, having its y-intercept at 5. Also, let $g(x) = f^{-1}(x)$ and let $h(x)$ have a graph that results from shifting the graph of $f(x)$ right 7 units and down 39 units. The functions $g(x)$ and $h(x)$ intersect once at $(12, 1)$. Find $a + b + c + d$.

a) 12 b) 36 c) 6 d) 24 e) 17 f) None of the other answers provided.

40) When the squares of two linear binomials are added, the result is a quadratic binomial. When their squares are subtracted, the result is a linear monomial. Which of the following statements cannot be true?

a) All presented answers are true.
b) Their difference is constant.
c) They are conjugates.
d) Their product is a difference of squares.
e) The square of their sum is a linear monomial.
f) None of the above

41) A ball is launched into the air from a height of over 20 meters. At times $1, 2$ and 3 seconds, the ball is at the heights $33, 30$ and 15 meters, respetively. If the flight path is parabolic before the ball strikes the ground, after approximately how many seconds will the ball strike the ground?

a) 4.2 seconds   b) 3.9 seconds   c) 5.8 seconds

d) 4.7 seconds   e) 3.6 seconds   f) None of the above

42) An isosceles triangle has sides measuring $\ln(32)$, $\ln(576)$ and $\dfrac{5}{\log_4\left(e^2\right)}$. Find the area of the triangle.

a) The triangle is not isosceles.

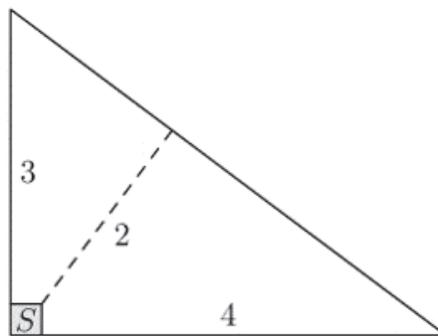b) Area $= (\ln 24)^2 \cdot \sqrt{\dfrac{5}{(\log_4\left(e^2\right))^2} - 1}$

c) Area $= 5\ln 24 \cdot \sqrt{\dfrac{1}{(\log_4\left(e^2\right))^2} - (\ln 24)^2}$

d) Area $= \ln 24 \cdot 16\sqrt{\dfrac{25}{(\log_4\left(e\right))^4} - (\ln 6)^2}$

e) Area $= \ln 24 \cdot \sqrt{\dfrac{25}{(\log_4\left(e^2\right))^2} - (\ln 24)^2}$

f) None of the above.

43) TIEBREAKER QUESTION: Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have length 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is not planted?

a) $\dfrac{13}{27}$ b) $\dfrac{5}{147}$ c) $\dfrac{23}{75}$ d) $\dfrac{2}{147}$ e) $\dfrac{2}{75}$